Application of IT in Mathematical Proofs and in Checking of Results of Pupils' Research

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Abstract: *In this paper attention is paid to several aspects of IT applications. Two main aspects: pupils' scientific research in mathematics and computer checking of the results obtained in them. The second aspect is the following one, namely sometimes with the help of a computer in the data generated purposefully it is possible to succeed in seeing a key for nontrivial mathematical proofs. The number of new results having been obtained by a computer researching combinatorial geometry shapes – polyominoes and tetrads have been given. In 2012 several computer programmes for designing of tetrads were developed and, thanks to those results, we managed to find an elegant proof, namely for every n $\geq$ 11 there is a full polyomino tetrad made of n-omino.*

Keywords: combinatorial geometry, polyomino, pentomino, tetrad, a computer-assisted proof.

Introduction

A polyomino is a connected plane figure formed of joining unit squares edge to edge, *a n-omino* is a polyomino consisting exactly of *n* squares, *a pentomino* is a polyomino consisting exactly of five squares. There are twelve pentominoes named F, I, L, N, P, T, U, V, W, X, Y, and Z respectively, see also Fig.1. The classic reference book on polyominoes is (Golomb, 1994).

Pentomino twins (or *p-twins*) are two equal polyominoes which can be assembled of pentominoes.

A **tetrad** is a plane figure made of four congruent shapes, joined so that each one shares a boundary (of a positive measure) with each. We can find some tetrads as well as information about the first contributors in the Martin Gardner's book (Gardner, 1989). One can find the next contribution in (http://userpages.monmouth.com/~colonel/tetrads/tetrads.html), (http://demonstrations.wolfram.com/Tetrads/). In accordance with (Gardner, 1989): "Michael R. W. Buckley, in the ***Journal of Recreational Mathematics***, 8 (1975), proposed the name **tetrad** for four simply connected planar regions, each pair of which shares a finite portion of a common boundary."

By a *computer-assisted proof* we mean a mathematical proof that has been at least partially generated by a computer. In 1976, the four colour theorem was the first major theorem to be verified using a computer programme.

The combinatorial geometry shapes – tetrads – have some connections with this famous problem, moreover, the very notion *tetrad* appeared almost at the same time when this famous problem was solved. An argument often being made against computer-aided proofs is that they lack mathematical elegance – that they provide no insights or new and useful concepts. In fact, this is an argument that could be advanced against any lengthy proof by exhaustion. Using combinatorial geometry shapes we show that a computer sometimes helps us to find a mathematically elegant proof, thus the mentioned argument is not always valid.

The world of polyominoes is rich in fascinating and easily understandable problems, for which no particular mathematical knowledge is necessary, but which are very far from being easily solved. We were able to obtain complete solutions (or proofs) of some of these problems by means of a computer, and it is unlikely that there could be some other short and easily way. One of the problems of such a type to which we will pay a special attention is the problem of finding the highest pentomino twins (two equal polyominoes which can be assembled of pentominoes). There are two basic guidelines in our selecting of problems: on the one hand, they must be sufficiently difficult or unsolved problems of mathematics, and, on the other hand, they must be understandable and accessible to students or pupils and appropriate as research topics for gifted pupils.

The problem of highest p-twins

The problem – *What are the highest p-twins*? – proposed by Andrejs Cibulis as a contest problem in the Latvian newspaper "Fokus", 1990. Only one solver was able to find twins with the height $H = 16$. Several pupils of Latvia have been investigated pentomino twins problems in their contest papers. Dmitrijs Hromakovs (Form 10, Riga Purvciems Secondary School) was able to find without a computer 227 *little twins* (10-ominoes two copies of which can be constructed from different pentominoes). The computer checking shows that there are 228 *little* p-twins (This pupil did not managed to find only one solution; later he together with his classmate found these last twins). This problem as one of the unsolved problem, appropriate as a topic of pupils' research, has also been proposed in (Cibulis, 2011). After having carefully examined the little twins one can find such twins that are useful as the blocks to construct the highest p-twins, see Fig. 3. A computer analysis shows that the highest p-twins are only those that can be made of the smaller p-twins, namely with the following structure: (4 + 2), (3 + 3), and (2 + 2 + 2), see Fig. 1-3. Let us mention that there is no uniqueness of building blocks, e. g.:

(FZWY + PL) = (TXUV + NI), (FWUV + TL) = (XYIN + PZ)

(NXW + ZUP) = (YTF + LIV), (FXY + VPL) = (VZN + UTI)

(LW + UY + TP) = (IN + VX + ZF), (LW + UY + TF) = (IN + VX + ZP).

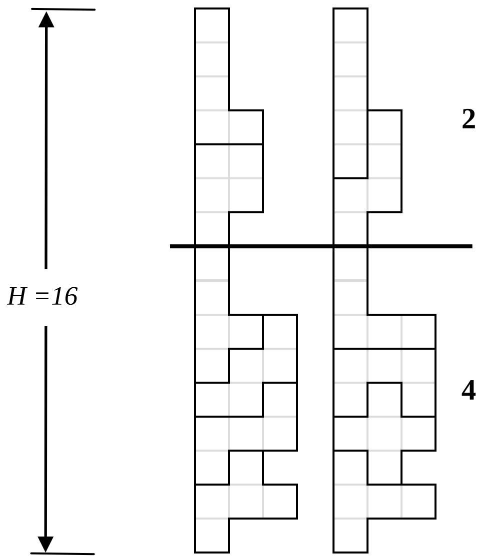


Fig. 1.

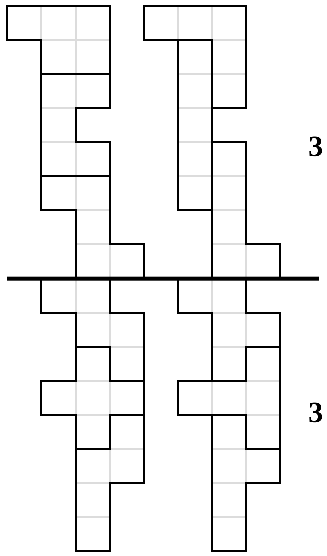


Fig. 2.

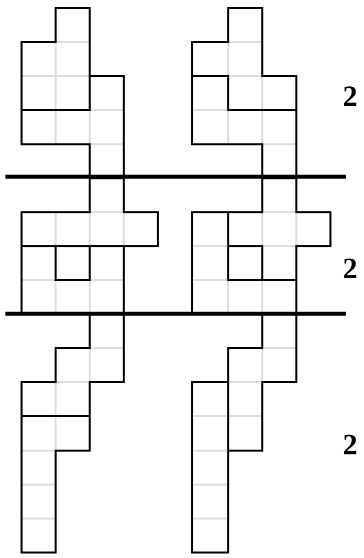


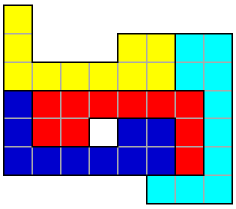
Fig. 3.

A short computer-assisted proof that there are no twins with H = 17 is as follows. First of all, 12 pentominoes were divided into two equal size sets. This can be done in 462 distinct ways. Using pentominoes of the one set “towers” were built recursively, in each step adding one pentomino in all possible ways. When "tower" was completed it was immediately tested whether it could be assembled with remaining 6 pentominoes. Few ideas were used to speed up the searching. In each step it is checked whether at least theoretically it is possible to have the height 17. The next optimising related to the order of 6 pentominoes. If the tower was built using pentominoes in some certain order then it is not necessary to check the reverse order, because it gives nothing new. To conclude that twins of the height 17 do not exist the programme worked approximately 6 hours. The idea to use complex numbers in coding polyominoes (see, e. g. (Rangel-Mondragón, 2005)) has been applied to solving this and other problems.

Tetrads

In this section we will formulate and prove the main result of this paper.

The pupils’ first research on tetrads in Latvia was done by Anastasija Jakovļeva in 2012 (Form 10, Riga Secondary School No. 92). She analysed polyomino tetrads. The main aim of her contest paper “Analysis of Tetrads” was to construct tetrads with prescribed properties. For example, find a tetrad having the square n by n as the single hole. To solve this problem it is necessary, and in fact also sufficiently, to find some tetrad with the unit hole. She succeeded in finding such a tetrad made from 10-ominoes, see Fig. 4. The next step is the applying of the well-known idea (each unit square can be replaced by the square 2×2), thus we have the so called p r o o f w i t h o u t w o r d s , see Fig. 4-5.



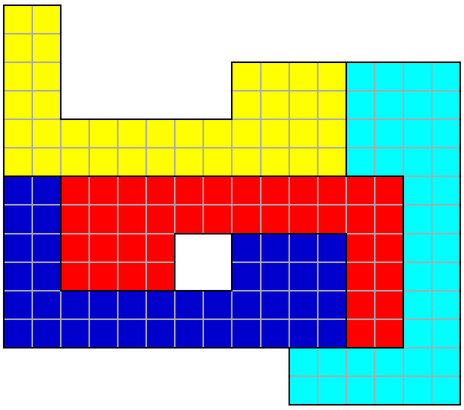


Fig. 4.   Fig. 5.

The most interesting result in her contest paper is tetrads that are transformable in the square 6 × 6, or in other words, she solved the problem: divide the square 6 × 6 into four equal parts and assemble a tetrad from them. Three partitions of the square are given in her contest paper, see Fig. 6. This result was checked by a computer. It turns out that there are no other solutions.

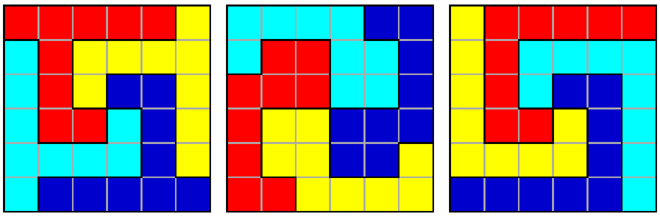


Fig. 6.

The construction of tetrads from these parts of the square would be an easy task and is left to the reader.

Remark. The minimal tetrad having the unit hole has been made from 9-ominoes. There are only two such minimal tetrads, see Fig. 7.

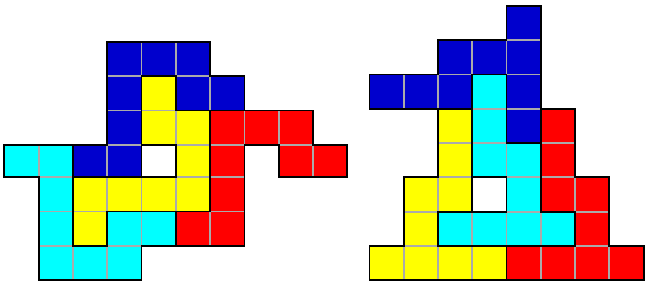


Fig. 7.

A short description of the algorithm for generating tetrads. At the beginning the full list of *n*-ominoes is generated, and then each polyomino is analysed separately. All the positions of the fixed polyomino are found (there may be up to 8 different positions). One polyomino is fixed. All possible ways to add the next polyomino are detected and they are stored in array. Then all the entries of an array of 3-subsets (the set that contains three elements) are considered. If the 3-subset has a property that every two polyominoes do not overlap and are side by side then the 3-subset together with the early fixed polyomino forms a tetrad. Checking all the 3-subsets can be accomplished with three cycles, but this step is slightly optimised. For example, checking the subsets $\{1; m; n\}$, $\{2; m; n\}$, ..., $\{m-1; m; n\}$, where $m < n$, it is not necessary to check many times whether m-th and n-th polyomino are not overlapping and adjacent. This step is optimised by using the principle of dynamic programming. In fact, this step is reduced to checking of adjacency and overlapping of each polyomino pair from the array. This algorithm seems to be efficient although several improvements might be made. In addition, the analysis of each pair is the most time extensive, so improvements of this step might bring time savings. Analysing polyominoes that are "far" from each other is redundant. Also more physical memory usage might lead to better results.

The number of tetrads made from *n*-ominoes is given in Table 1. As far as we know the number of tetrads have determined for the first time. Let us mention that computer search for tetrads made from 16-ominoes took approximately 115 hours. Calculations were performed on the oldish HP DV8075 laptop with 2.2 GHz AMD processor and 1 GB RAM. Programme was implemented in Free Pascal.

Table 1

Polyomino tetrads

n	**Number of suitable *n*-ominoes**	**Number of tetrads**
8	8	14
9	42	83
10	187	341
11	739	1388
12	2871	5648
13	11300	22688
14	44440	90243
15	172984	352243
16	670107	1373595

Theorem. For each $n \geq 11$ there is a *n*-omino that forms a tetrad without holes.

The very idea of this theorem is simple. It is necessary to find such a tetrad that can be enlarged to the next one. It is easily to propose such an idea, but it is hard to find such a tetrad. The problem itself of finding small full tetrads is not an easy task, e. g, the *manual* finding of the full tetrad consisting of 11-omino can require several hours of work. Let us emphasise that the key how to prove this theorem was found looking through the huge set of tetrads generated by the computer. The necessary tetrad we found is made from 17-omino.

Proof. The full six tetrads made from *n*-omino, $11 \leq n \leq 16$, are shown in Fig. 8.

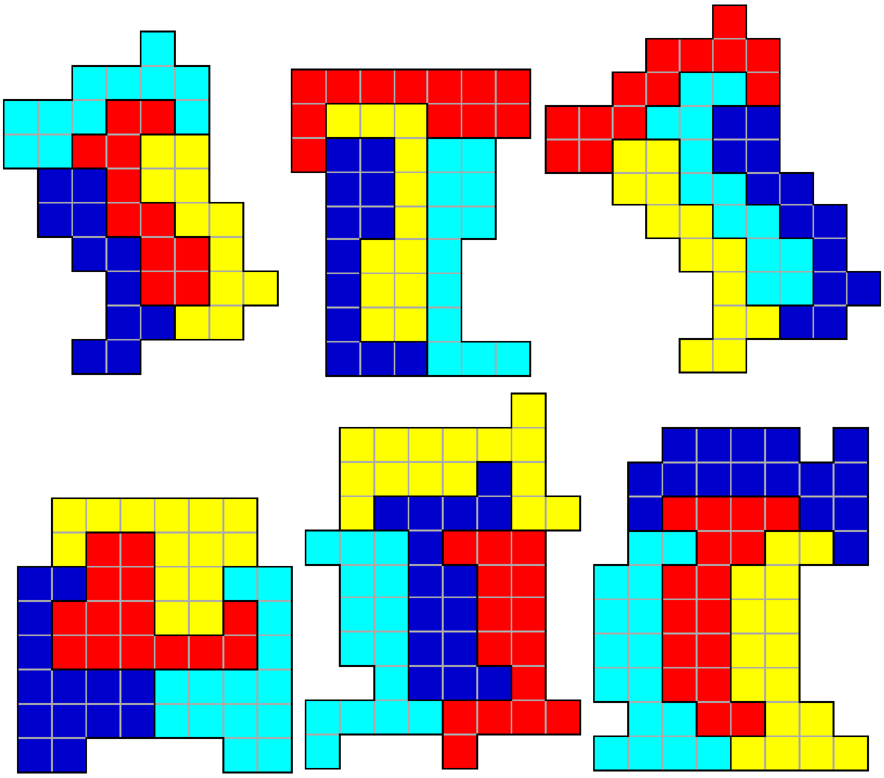


Fig. 8. **Full *n*-omino tetrads for *n* = 11, 12, 13, 14, 15, and 16.**

The key tetrad made from 17-omino is shown in Fig. 9.

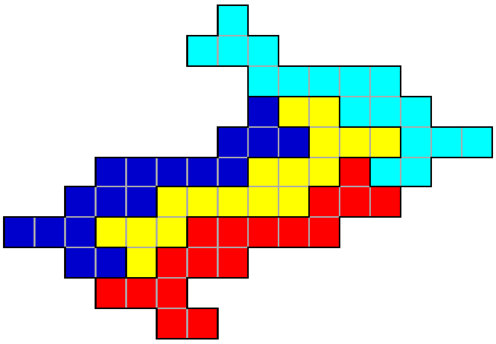


Fig. 9.

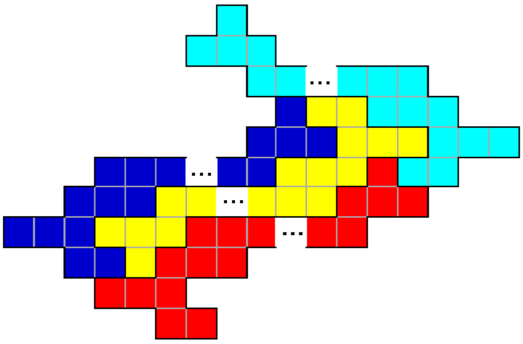


Fig. 10.

Carefully looking at this tetrad one can observe that it can be stretched (extended) arbitrarily long in the horizontal direction, as shown in Fig. 10, thus the theorem has been proved.

Minimal nets

Let $T(n)$ be the smallest number of I-trominoes (1×3) required to stop any more being placed on the board $n \times n$. Each tromino must line up with the squares on the board, so that it covers exactly three squares. The first nine values of the numbers $T(n)$ for $3 \leq n \leq 11$ are as follows: 3, 4, 5, 7, 9, 13, 16, 20, 24. Corresponding nine nets are given in Fig.11.

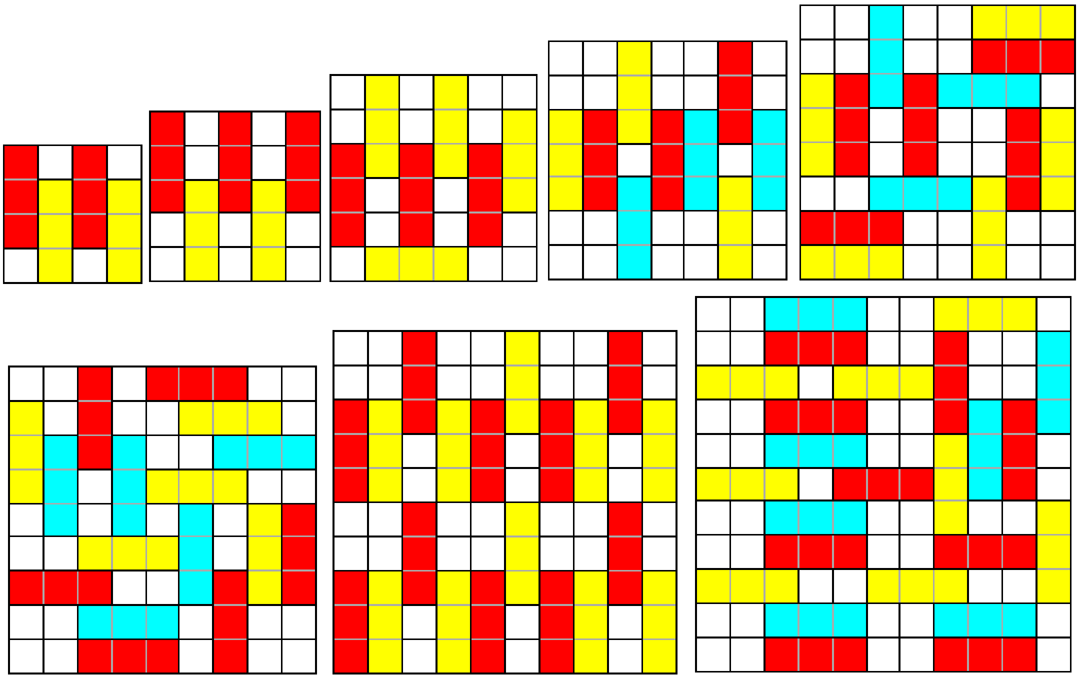


Fig. 11. **Minimal I-tromino nets.**

H y p o t h e s i s : The minimum number $T(n)$ satisfies the inequality

$$\left\lfloor \frac{n^2}{5} \right\rfloor - 1 \leq T(n) \leq \left\lceil \frac{n^2}{5} \right\rceil + 1.$$

Conclusion

The lack of efficient algorithms creates serious difficulties in constructing tetrads. Nowadays a computer plays an important role in mathematics. Firstly, IT applications provide an opportunity to verify pupils' results in scientific research in mathematics. Secondly, a computer allows experimenting and the results give a more profound understanding of the problem and induce new ideas and insights. Since computers become highly sophisticated and more powerful new horizons are revealing constantly. We expect that partially or completely computer-assisted proofs will occur in the future more often. It also has a negative effect, because the programmes are often bulky and difficult to verify. The problem of investigating the numbers $T(n)$ could serve as the challenging and appropriate topic for pupils' and students' project works and research. This sequence has not been included in the famous Encyclopaedia (http://oeis.org), at least not yet.

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