INTEGRATION OF SOME RATIONAL AND IRRATIONAL FUNCTIONS USING SOFTWARE "MATHCAD"

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Abstract: *One of the most important classes of elementary functions, whose integrals can be found in comparatively simple way and always are elementary functions, are rational functions. Except for a few very special cases, currently we have no way to find the integral of a general rational function. In practicality, the method we shall develop is long and cumbersome, but the most important thing is that, in general, it will work. Before we start with the integration, we need to develop a method of reducing a rational function called the method of partial fractions. For evaluating some particular types of irrational functions our endeavour will be to arrive at a rational function through an appropriate substitution. We motivate our actions with the examples. Mathcad is one of popular computer algebra system. Math software and technology are helpful aide in teaching of Maths. Although for most problems we meet in teaching of Math it is enough to use computation ability, in certain cases the program function becomes necessary and very helpful.*

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Introduction

Mathcad is studied in the course of higher mathematics. The most important in the course is to consider more complicated themes, for example, multiple integrals or differential equations. It is quite simple operation in Mathcad "find the indefinite integral symbolically". So usually students consider only the indefinite integral symbol and few examples about this part of higher mathematics.
The lecturer does not have enough time for explanation about possibilities to evaluate the indefinite integral in Mathcad. More time usually spend to speak about definite integrals and multiple integrals. If we consider multiple integrals or solve differential equations we have indefinite integrals which Mathcad cannot calculate or solutions are complicated, for example, using special functions.

Integration of rational functions.

A rational function is a function which is the quotient of two polynomials. Any rational function (a ratio of polynomials) $\frac{P_n(x)}{R_m(x)}$ by expressing it as sum of polynomial $Q(x)$ and partial fractions

$$\frac{A}{x+a}, \frac{B}{(x+a)^k}, \frac{Ax+B}{x^2+px+q}, \frac{Ax+B}{(x^2+px+q)^k}.$$

Consider integrals of the type $\int \frac{P(x)}{x^2+px+q}dx$ where $P(x)$ is a polynomial, $p,q \in R$. If the degree of the polynomial $P(x)$ is greater than 1, the division of $P(x)$ by x^2+px+q results in a polynomial $Q(x)$ and a polynomial $ax+b$, as the remainder. Consequently $\frac{P(x)}{x^2+px+q} = Q(x) + \frac{ax+b}{x^2+px+q}$.
The integration of the polynomial $Q(x)$ does not present any difficulties and hence the problem reduces to integrating a fraction $\frac{ax+b}{x^2+px+q}$, if $a^2+b^2 \neq 0$.
If $x^2+px+q = (x-x_1)(x-x_2)$, where x_1 and x_2 are two different real numbers, then there exist real constants *A* and *B* such that $\frac{ax+b}{x^2+px+q} = \frac{A}{x-x_1} + \frac{B}{x-x_2}$, what is useful for integration. Unknown constants *A* and *B* are found by "the method of indefinite coefficients".
Example 1. Calculate $\int \frac{1}{x^2+px+q}dx$, where $p,q \in R$.
We consider the case, that the polynomial x^2+px+q has a double root or is positive on an interval. Using software "Mathcad" for calculating this examples students have not problem with solutions. The result is given in easy to use form.

$$\int \frac{1}{x^2+2\cdot x+1}dx \rightarrow -\frac{1}{x+1}$$

or

$$\int \frac{1}{x^2 + 4\cdot x + 8}\,dx \rightarrow \frac{\operatorname{atan}\left(\frac{x}{2}+1\right)}{2}$$

Remark. Remember that the computer algorithm might simplify expressions different from what we are used to. Also, it does not give the integration constant "$+C$".

If the polynomial $x^2 + px + q = (x - x_1)(x - x_2)$, where x_1 and x_2 are two different real numbers, then there exist real constants *A* and *B* such that $\frac{ax+b}{x^2+px+q} = \frac{A}{x-x_1} + \frac{B}{x-x_2}$, what is useful for integration. It is important for student to know about integration methods "the method of indefinite coefficients".

Example 2. Calculate the indefinite integral $\int \frac{dx}{x^2-3x+2}$.

The polynomial $x^2 - 3x + 2$ has two different roots $x_1 = 1$, $x_2 = 2$ (both the real numbers). The result Mathcad gives is

$$\int \frac{1}{x^2 - 3\cdot x + 2}\,dx \rightarrow -2\cdot \operatorname{atanh}(2\cdot x - 3)$$

The integration result has inverse hyperbolic tangent. This result is difficult for understanding for students. Almost all students do not know about the inverse hyperbolic tangent.

This rational function $\frac{1}{x^2-3x+2}$ we can be written in the form of the partial fractions decomposition and then integrate new expressions. The result is easier to understand for students and to continue work with multiple integrals or differential equations.

$$\int \frac{dx}{x^2 - 3x + 2} = \int \left(\frac{1}{x-2} - \frac{1}{x-1}\right) dx = \ln|x-2| - \ln|x-1| + C$$

The similar result we give from Mathcad result using the formula

$$\operatorname{arctanh} x = \frac{1}{2}\ln\left|\frac{x+1}{x-1}\right|,$$

$$-2\operatorname{arctanh}|2x-3| = -\ln\left|\frac{2x-2}{2x-4}\right| = -\ln\left|\frac{x-1}{x-2}\right| = \ln|x-2| - \ln|x-1|$$

These transformations require additional knowledge for students.

Using Mathcad functions (simplify, expand, factor, parfrac etc.) sometimes we can get the better result.

$$\int \frac{1}{x^2 - 3\cdot x + 2}\,dx \rightarrow -2\cdot \operatorname{atanh}(2\cdot x - 3) \text{ simplify } \rightarrow \ln(4 - 2\cdot x) - \ln(2\cdot x - 2)$$

Working with Mathcad students can at first find a partial fractions decomposition for a given rational function (the Mathcad function parfrac) and then evaluate indefinite integrals.

$$f(x) := \frac{1}{x^2 - 3x + 2} \text{ parfrac } \rightarrow \frac{1}{x-2} - \frac{1}{x-1}$$

$$\int f(x)\,dx \rightarrow \ln(x-2) - \ln(x-1)$$

Example 3. Calculate the indefinite integral $\int \frac{x\,dx}{x^2+x+1}$.

The result Mathcad gives is

$$\int \frac{x}{x^2 + x + 1}\,dx \rightarrow \frac{\ln\left[\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}\right]}{2} - \frac{\sqrt{3}\cdot\left[2\cdot\operatorname{atan}\left[\frac{2\cdot\sqrt{3}\cdot\left(x+\frac{1}{2}\right)}{3}\right] - \pi\right]}{6}$$

In this examples it is difficult to understand for students the inverse hyperbolic tangent and $-\pi$ as a part of constant C.

The examples were with polynomial with 2nd degree, in the similar way can calculate the indefinite integrals of rational functions $\int \frac{P(x)}{R(x)} dx$ where $P(x), R(x)$ are the polynomials in *x*.

Example 4. Calculate the indefinite integral $\int \frac{11x^3-107x+108}{x^4+x^3-30x^2+76x-56} dx$.
If students integrate rational function as given in Mathcad it takes more time to get the result.

$$\int \frac{11x^3 - 107x + 108}{x^4 + x^3 - 30x^2 + 76x - 56} dx \rightarrow 7 \cdot \ln(x-2) + 4 \cdot \ln(x+7) - \frac{3 \cdot x - 7}{x^2 - 4 \cdot x + 4}$$

The similar result can get faster if at first students use the method of partial fractions and then integrate these fractions.

$$f(x) := \frac{11x^3 - 107x + 108}{x^4 + x^3 - 30x^2 + 76x - 56} \text{ parfrac } \rightarrow \frac{7}{x-2} + \frac{3}{(x-2)^2} - \frac{2}{(x-2)^3} + \frac{4}{x+7}$$

$$\int f(x)\, dx \rightarrow 7 \cdot \ln(x-2) + 4 \cdot \ln(x+7) - \frac{3 \cdot x - 7}{x^2 - 4 \cdot x + 4}$$

Integration of irrational functions.

The task of integrating functions gets tougher if the given function is an irrational one, that is, it is not of the form $\frac{P(x)}{R(x)}$. For evaluating some particular types of irrational functions our endeavour will be to arrive at a rational function through an appropriate substitution.
Integration of functions containing only fractional powers of *x*. In this case we put $x = t^n$, where is the lowest common multiple of the denominators of powers of *x*. This substitution reduces the function to a rational function of *t*.

Example 5. Calculate the indefinite integral $\int \frac{1}{\sqrt{x}+\sqrt[3]{x}} dx$.

Using Mathcad sometimes it is important to write $\sqrt[n]{x}$ in the form $x^{\frac{1}{n}}$. In this example Mathcad does not give the result

$$\int \frac{1}{\sqrt{x}+\sqrt[3]{x}} dx \rightarrow \int \frac{1}{\sqrt{x}+x^{\frac{1}{3}}} dx \qquad \text{or} \qquad \int \frac{1}{\sqrt{x}+x^{\frac{1}{3}}} dx \rightarrow \int \frac{1}{\sqrt{x}+x^{\frac{1}{3}}} dx$$

To calculate the indefinite integral $\int \frac{1}{\sqrt{x}+\sqrt[3]{x}} dx$ students can use the substitution. It is important to choose correct substitution. Partially students can do this using Mathcad:

$$f(x) := \frac{1}{\sqrt{x}+\sqrt[3]{x}} \qquad \text{Substitution:} \qquad x = t^6$$

$$y(t) := f(x) \text{ substitute}, \sqrt{x} = t^3, \sqrt[3]{x} = t^2 \rightarrow \frac{1}{t^2 \cdot (t+1)}$$

$$dy(t) := \frac{d}{dt} t^6 \text{ simplify } \rightarrow 6 \cdot t^5$$

$$h(t) := y(t) \cdot dy(t) \rightarrow \frac{6 \cdot t^3}{t+1}$$

$$\text{rez1}(t) := \int h(t)\, dt \rightarrow \frac{3 \cdot t^4 \cdot \text{hypergeom}(1,4,5,-t)}{2}$$

On the one hand the integration result contains the hypergeometric function. This result is difficult for understanding for students. On the other hand the resulting function $h(t)$ is a rational function of t and for this function students can use the method of partial fractions and then integrate these fractions. Then the result is given in easy to use form:

$$h1(t) := h(t) \text{ parfrac } \rightarrow 6 \cdot t^2 - 6 \cdot t - \frac{6}{t+1} + 6$$

$$rez2(t) := \int h1(t)\, dt \rightarrow 6 \cdot t - 6 \cdot \ln(t+1) - 3 \cdot t^2 + 2 \cdot t^3$$

$$rez(x) := rez2(t) \text{ substitute}, t = \sqrt[6]{x} \rightarrow 2 \cdot \sqrt{x} - 6 \cdot \ln\left(x^{\frac{1}{6}} + 1\right) - 3 \cdot x^{\frac{1}{3}} + 6 \cdot x^{\frac{1}{6}}$$

Integration of $\frac{1}{(x+e)\sqrt{ax^2+bx+c}}$ is used the substitution $x + e = \frac{1}{t}$, and then simplify the expression.

<u>Example 6.</u> Calculate the indefinite integral $\int \frac{1}{(x+1)\sqrt{x^2+4x+2}} dx$.

In this example Mathcad also does not show the result if the students want to write as given

$$\int \frac{1}{(x+1) \cdot \sqrt{x^2 + 4x + 2}} dx \rightarrow \int \frac{1}{(x+1) \cdot \sqrt{x^2 + 4 \cdot x + 2}} dx$$

Having applied the additional knowledge about substitution students can use Mathcad possibility for getting resulting function $h(t)$:

$$f(x) := \frac{1}{(x+1) \cdot \sqrt{x^2 + 4 \cdot x + 2}} \qquad \text{Substitution:} \quad x + 1 = \frac{1}{y}$$

$$a(t) := x + 1 = \frac{1}{t} \text{ solve}, x \rightarrow \frac{1}{t} - 1 \qquad b(x) := x + 1 = \frac{1}{t} \text{ solve}, t \rightarrow \frac{1}{x+1}$$

$$y(t) := f(x) \text{ substitute}, x = \frac{1}{t} - 1 \rightarrow \frac{t}{\sqrt{\frac{2 \cdot t - t^2 + 1}{t^2}}}$$

$$dy(t) := \frac{d}{dt} a(t) \rightarrow -\frac{1}{t^2}$$

$$h(t) := y(t) \cdot dy(t) \rightarrow -\frac{1}{t \cdot \sqrt{\frac{2 \cdot t - t^2 + 1}{t^2}}}$$

$$\int h(t)\, dt \rightarrow -\int \frac{\frac{1}{t}}{\sqrt{\frac{2}{t} + \frac{1}{t^2} - 1}} dt$$

Mathcad could not simplify it fully. If student having applied knowledge about the expression $h(t)$ transformation then the Mathcad get the integration result

$$\mathrm{result1}(t) := \int \frac{-1}{\sqrt{2t - t^2 + 1}}\, dt \rightarrow -\mathrm{asin}\left[\frac{1}{5}\cdot\sqrt{5}\cdot(t-2)\right]$$

$$\mathrm{result}(x) := \mathrm{result1}(t)\ \mathrm{substitute}, t = b(x) \rightarrow -\mathrm{asin}\left[-\frac{\sqrt{5}\cdot(2\cdot x+1)}{5\cdot x+5}\right]$$

Conclusion

Software "Mathcad" is very convenient for computing integrals of rational or irrational functions.

Integration of an elementary function not always leads to an elementary function. As shown in examples there exist elementary functions whose integrals are inexpressible in terms of elementary functions without substitution.

The method of partial fractions is used not only for finding integrals, but also for analyzing linear differential systems like resonant circuits and feedbackcontrol systems (electrical or mechanical engineering).

For receiving results in Mathcad in convenient form students are required theoretical knowledge about integration of some rational and irrational functions.

The lecturer can suggest to students to use other software: "Matlab", "Mathematica", but these software are more complicated and require more time for obtaining skills for work.

Software "Mathcad" advantage is the possibility easy and quickly write the received results.

References

Lewin, J., 2003. An Interactive Introduction to Mathematical Analysis. – Cambridge University Press.

Kitjanov, D., 2006. Mathcad 13. – Spb: BXV-Peterburg.